

# ON GENERAL LAGRANGIAN FORMULATIONS FOR ARBITRARY MIXED-SYMMETRIC HIGHER-SPIN FERMIONIC FIELDS ON MINKOWSKI BACKGROUNDS

A.A. Reshetnyak

Laboratory of Non-Linear Media Physics, Institute of Strength Physics and Materials Science, Tomsk, 634021, Russia.

E-mail: reshet@ispms.tsc.ru, reshet@tspu.edu.ru

The details of unconstrained Lagrangian formulations (being continuation of earlier developed research for Bose particles in NPB 862 (2012) 270, [arXiv:1110.5044[hep-th]], Phys. of Part. and Nucl. 43 (2012) 689, [arXiv:1202.4710 [hep-th]]) are reviewed for Fermi particles propagated on an arbitrary dimensional Minkowski space-time and described by the unitary irreducible half-integer higher-spin representations of the Poincare group subject to Young tableaux  $Y(s_1, \dots, s_k)$  with  $k$  rows. The procedure is based on the construction of the Verma modules and finding auxiliary oscillator realizations for the orthosymplectic  $osp(k|2k)$  superalgebra which encodes the second-class operator constraints subsystem in the HS symmetry superalgebra. Applying of an universal BRST-BFV approach permit to reproduce gauge-invariant Lagrangians with reducible gauge symmetries describing the free dynamics of both massless and massive fermionic fields of any spin with appropriate number of gauge and Stukelberg fields. The general construction possesses by the obvious possibility to derive Lagrangians with only holonomic constraints.

**Keywords:** higher spins; BRST symmetry; Lagrangian formulation; Verma module; gauge invariance

## 1 Introduction

The interest to higher-spin (HS) field theory is based on the hopes to reconsider the problems of an unique description of variety of elementary particles and all known interactions, in particular, due to recent success with relating to finding of Higgs boson on LHC[1]. One should remind, that it waits, in addition, both the proof of supersymmetry display, and probably a new insight on origin of Dark Matter ([2]). Due to close interrelation of HS field theory to superstring theory, which operates with an infinite tower of HS fields with integer and half-integer spins it can be viewed as an method to study a superstring theory structure. On current state of HS field theory one may know from the reviews [3], [4], [5], [6]. The paper considers the results of constructing Lagrangian formulations (LFs) for free half-integer both massless and massive mixed-symmetry spin-tensor HS fields on flat  $\mathbb{R}^{1,d-1}$ -space-time subject to arbitrary Young tableaux (YT)  $Y(n_1, \dots, n_k)$  for  $s_1 = n_1 + \frac{1}{2}, \dots, s_k = n_k + \frac{1}{2}$  in Fronsdal metric-like formalism on a base of BFV-BRST approach [7], and precesses the results which appear soon in [10] (as continuation of the research for arbitrary HS fields with integer spin made in [8]).

We know that for higher then  $d = 4$  space-time dimensions, there appear, in addition to totally symmetric irreducible representations of Poincare or (Anti)-de-Sitter ((A)dS) algebras the mixed-symmetry representations determined by more than one spin-like pa-

rameters [11], [12], [13]. Whereas for the former ones the LFs both for massless and massive free higher-spin fields is well enough developed [14], [15], [16], [17], [18] [19] as well as on base of BFV-BRST approach, e.g. in [20]–[22], [23], for the latter the problem of their field-theoretic description has not yet solved. So, the main result within the problem of constrained LF for arbitrary massless mixed-symmetry spin-tensor HS fields on a Minkowski space-time was obtained in [24] in so-called "frame-like" formulation (in AdS space in [25]), whereas in the "metric-like" formulation corresponding unconstrained Lagrangians were derived in closed manner for only reducible Poincare group  $ISO(1, d-1)$  representations in [26] (see as well, [27]).

We use, first, the conventions for the metric tensor  $\eta_{\mu\nu} = \text{diag}(+, -, \dots, -)$ , with Lorentz indices  $\mu, \nu = 0, 1, \dots, d-1$ , second, the relations  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ , for Dirac matrices  $\gamma^\mu$ , third, the notation  $\epsilon(A)$ ,  $gh(A)$  for the respective values of Grassmann parity and ghost number of a quantity  $A$ , and denote by  $[A, B]$  the supercommutator of quantities  $A, B$ , which for theirs definite values of Grassmann parities is given by  $[A, B] = AB - (-1)^{\epsilon(A)\epsilon(B)}BA$ .

## 2 Half-Integer HS Symmetry Algebra for Fermionic fields

A massless half-integer spin Poincare group irrep in  $\mathbb{R}^{1,d-1}$  is described by rank  $\sum_{i \geq 1}^k n_i$  spin-

tensor field  $\Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} \equiv \Psi_{\mu_1^1 \dots \mu_{n_1}^1, \mu_1^2 \dots \mu_{n_2}^2, \dots, \mu_1^k \dots \mu_{n_k}^k} A(x)$  with generalized spin  $\mathbf{s} = (n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, \dots, n_k + \frac{1}{2})$ ,  $(n_1 \geq n_2 \geq \dots \geq n_k > 0, k \leq [(d-1)/2])$  subject to a YT with  $k$  rows of lengths  $n_1, n_2, \dots, n_k$  and suppressed Dirac index  $A$ ,

$$\Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} \leftrightarrow \begin{array}{|c|c|c|c|c|c|} \hline \mu_1^1 & \cdot & \cdot & \cdot & \cdot & \mu_{n_1}^1 \\ \hline \mu_1^2 & \cdot & \cdot & \cdot & \mu_{n_2}^2 & \\ \hline \cdot & \cdot & \cdot & \cdot & & \\ \hline \mu_1^k & \cdot & \cdot & \mu_{n_k}^k & & \\ \hline \end{array} . \quad (1)$$

The spin-tensor is symmetric with respect to the permutations of each type of indices  $\mu^i$  and obeys to the Dirac (2), gamma-traceless (3) and mixed-symmetry equations (4) [for  $i, j = 1, \dots, k$ ;  $l_i, m_i = 1, \dots, n_i$ ]:

$$\iota \gamma^\mu \partial_\mu \Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} = 0, \quad (2)$$

$$\gamma^{\mu_i} \Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} = 0, \quad (3)$$

$$\Psi_{(\mu^1)_{n_1}, \dots, \{(\mu^i)_{n_i}, \dots, \underbrace{\mu_{l_i}^j \dots \mu_{l_j}^j}_{\dots \mu_{n_j}^j} \dots \mu_{n_j}^j, \dots, (\mu^k)_{n_k}} = 0 \quad (4)$$

for  $i < j$ ,  $1 \leq l_j \leq n_j$  and where the bracket below denote that the indices in it do not include in symmetrization.

Joint description of all half-integer spin  $ISO(1, d-1)$  group irreps can be standardly reformulated with an auxiliary Fock space  $\mathcal{H}$ , generated by  $k$  pairs of bosonic creation  $a_{\mu^i}^i(x)$  and annihilation  $a_{\nu^j}^{j+}(x)$  operators (in symmetric basis),  $i, j = 1, \dots, k$ ,  $\mu^i, \nu^j = 0, 1, \dots, d-1$ :  $[a_{\mu^i}^i, a_{\nu^j}^{j+}] = -\eta_{\mu^i \nu^j} \delta^{ij}$  and a set of constraints for an arbitrary string-like (so called *basic*) vector  $|\Psi\rangle \in \mathcal{H}$ , being as well Dirac spinor,

$$|\Psi\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \dots \sum_{n_k=0}^{n_{k-1}} \Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} \times \prod_{i=1}^k \prod_{l_i=1}^{n_i} a_i^{+\mu_{l_i}^i} |0\rangle, \quad (5)$$

$$(\tilde{t}_0, \tilde{t}^i, t^{i_1 j_1}) |\Psi\rangle = 0, \text{ for } i \leq j; i_1 < j_1, \quad (6)$$

where  $(\tilde{t}_0, \tilde{t}^i, t^{i_1 j_1}) = (-i\gamma^\mu \partial_\mu, \gamma^\mu a_\mu^i, a_\mu^{i_1+} a_\mu^{j_1 \mu})$ .

The set of  $(\frac{1}{2}k(k+1) + 1)$  primary constraints (6),  $\{o_\alpha\} = \{\tilde{t}_0, \tilde{t}^i, t^{i_1 j_1}\}$ , with additional condition,  $g_0^i |\Psi\rangle = (n_i + \frac{d}{2}) |\Psi\rangle$  for number particles operators,  $g_0^i = -a_\mu^{i+} a^{\mu i} + \frac{d}{2}$ , are equivalent to Eqs. (2)–(4) for given spin  $\mathbf{s}$ .

The fermionic nature of equations (2), (3) and the bosonic one of the primary constraint operators  $\tilde{t}_0, \tilde{t}^i$  with respect to the standard Lorentz-like Grassmann parity,  $\epsilon(\tilde{t}_0) = \epsilon(\tilde{t}^i) = 0$  are in contradiction and resolve the problem of,  $(\tilde{t}^i)^2 = \frac{1}{2} \gamma^\mu \gamma^\nu a_\mu^i a_\nu^i + a_\nu^i a_\mu^i =$

$a_\mu^i a_\mu^i = 2l^{ii}$ , with new "traceless" operator, we equivalently transform above operators into fermionic ones. Following to Ref. [21], [22] we introduce a set of  $(d+1)$  Grassmann-odd gamma-matrix-like objects  $\tilde{\gamma}^\mu, \tilde{\gamma}$ , subject to

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}, \quad \{\tilde{\gamma}^\mu, \tilde{\gamma}\} = 0, \quad \tilde{\gamma}^2 = -1, \quad (7)$$

and related to the conventional gamma-matrices as:  $\gamma^\mu = \tilde{\gamma}^\mu \tilde{\gamma}$ .

Therefore, the odd constraints,

$$t_0 = -i\tilde{\gamma}^\mu \partial_\mu, \quad t^i = \tilde{\gamma}^\mu a_\mu^i, \quad (8)$$

related to the operators (6) as:  $(t_0, t^i) = \tilde{\gamma}(\tilde{t}_0, \tilde{t}^i)$ , solve the problem above.

The finding of Lagrangian as,  $\mathcal{L} \sim \langle \Psi | Q | \Psi \rangle$ , implies the Hermiticity of BFV-BRST operator  $Q$ ,  $Q = C^\alpha o_\alpha + \dots$ , that means the extension of the set  $\{o_\alpha\}$  up to one of  $\{o_I\} = \{o_\alpha, l_i, l_{ij}; o_\alpha^+, l_i^+, l_{ij}^+, g_0^i\}$ , for divergent and gradient operators  $(l_i, l_{ij}^+) = -i(a_\mu^i \partial_\mu, a_\mu^{i+} \partial_\mu)$  and for  $l_{ij}^+ = \frac{1}{2} a_i^{\mu+} a_j^{\mu+}$ ,  $i \leq j$ , which is closed with respect to supercommutator multiplication  $[\ , \ ]$  and hermitian conjugation related to odd scalar product on  $\mathcal{H}$ ,

$$\begin{aligned} \langle \tilde{\Phi} | \Psi \rangle &= \int d^d x \sum_{(n)_k=0}^{\infty, (n)_{k-1}} \sum_{(p)_k=0}^{\infty, (p)_{k-1}} \langle 0 | \prod_{j=1; m_j=1}^{k; p_j} a_j^{\nu_j^{m_j}} \\ &\times \Phi_{(\nu^1)_{p_1}, (\nu^2)_{p_2}, \dots, (\nu^k)_{p_k}} \tilde{\gamma}_0 \Psi_{(\mu^1)_{n_1}, (\mu^2)_{n_2}, \dots, (\mu^k)_{n_k}} \\ &\times \prod_{i=1; l_i=1}^{k; n_i} a_i^{+\mu_{l_i}^i} |0\rangle, \end{aligned} \quad (9)$$

for  $\sum_{(n)_k=0}^{\infty, (n)_{k-1}} \equiv \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \dots \sum_{n_k=0}^{n_{k-1}}$ ,  $n_i, p_j \in \mathbb{N}_0$ . Operators  $o_I$  satisfy to the Lie-superalgebra commutation relations,  $[o_I, o_J] = f_{IJ}^K o_K$ , for structure constants  $f_{IJ}^K = -(-1)^{\epsilon(o_I)\epsilon(o_J)} f_{JI}^K$ , to be determined by anticommutators,

$$\begin{aligned} \{t_0, t_0\} &= -2l_0, \quad \{t_0, t_i\} = 2l_i, \quad \{t_i, t_j\} = 4l_{ij}, \\ \{t_i, t_j^+\} &= 2(-g_0^i \delta_{ij} + t_{ji} \theta^{ij} + t_{ij}^+ \theta^{ji}), \end{aligned} \quad (10)$$

(with Heaviside  $\theta$ -symbol  $\theta^{ij}$ ) and from the multiplication table 1 with only commutators.

The products  $B_{i_1 j_1}^{i_2 j_2}, A_{i_1 j_1}^{i_2 j_2, i_1 j_1}, F_{i_1 j_1, i}^{i_2 j_2, i_1 j_1}, L_{i_2 j_2, i_1 j_1}^{i_2 j_2, i_1 j_1}$  in the table 1 are given by the relations,

$$\begin{aligned} B_{i_1 j_1}^{i_2 j_2} &= (g_0^{i_2} - g_0^{j_2}) \delta_{i_1}^{i_2} \delta_{j_1}^{j_2} + (t_{j_1}^{j_2} \theta^{j_2}_{j_1} + t_{j_1}^{j_2+} \theta_{j_1}^{j_2}) \delta_{i_1}^{i_2} - (t_{i_1}^{i_2} \theta^{i_2}_{i_1} + t_{i_1}^{i_2+} \theta_{i_1}^{i_2}) \delta_{j_1}^{j_2}, \\ A_{i_2 j_2, i_1 j_1}^{i_2 j_2, i_1 j_1} &= t^{i_1 j_2} \delta^{i_2 j_1} - t^{i_2 j_1} \delta^{i_1 j_2}, \quad F_{i_2 j_2, i}^{i_2 j_2, i_1 j_1} = t^{[i_2 j_2} \delta^{j_2] i}, \\ L_{i_2 j_2, i_1 j_1}^{i_2 j_2, i_1 j_1} &= \frac{1}{4} \{ \delta^{i_2 i_1} \delta^{j_2 j_1} [2g_0^{i_2} \delta^{i_2 j_2} + g_0^{j_2} + g_0^{j_2}] \\ &\quad - (\delta^{j_2 i_1} [t_{j_1}^{i_2} \theta^{i_2 j_1} + t^{i_2 j_1}] + \theta^{j_1 i_2}] + (j_2 \leftrightarrow i_2) \}. \end{aligned} \quad (11)$$

$[\downarrow, \rightarrow]$	$t^{i1j1}$	$t_{i1j1}^+$	$l_0$	$l^i$	$l^{i+}$	$l^{i1j1}$	$l^{i1j1+}$	$g_0^i$
$t_0$	0	0	0	0	0	0	0	0
$t^{i2}$	$-t^{j1}\delta^{i2j1}$	$-t_{i1}\delta^{i2}_{j1}$	0	0	$-t_0\delta^{i2i}$	0	$-\frac{1}{2}t\{i1+\delta j1\}^{i2}$	$t^{i2}\delta^{i2i}$
$t^{i2+}$	$t^{i1}+\delta^{i2j1}$	$t_{j1}^+\delta_{i1}^{i2}$	0	$t_0\delta^{i2i}$	0	$\frac{1}{2}t\{i1\delta j1\}^{i2}$	0	$-t^{i2}+\delta^{i2i}$
$t^{i2j2}$	$A^{i2j2,i1j1}$	$B^{i2j2}_{i1j1}$	0	$l^{j2}\delta^{i2i}$	$-l^{i2}+\delta^{j2i}$	$l\{j1j2\delta^{i1}\}^{i2}$	$-l^{i2}\{i1+\delta j1\}^{j2}$	$F^{i2j2,i}$
$t_{i2j2}^+$	$-B^{i1j1}_{i2j2}$	$A_{i1j1,i2j2}^+$	0	$l_{i2}\delta_{j2}^i$	$-l_{j2}^+\delta_{i2}^i$	$l_{i2}\{j1\delta_{j2}^{i1}\}$	$-l_{j2}\{j1+\delta_{i2}^{i1}\}$	$-F_{i2j2}^{i+}$
$l_0$	0	0	0	0	0	0	0	0
$l^j$	$-lj1\delta^{i1j}$	$-l_{i1}\delta_{j1}^j$	0	0	$l_0\delta^{ji}$	0	$-\frac{1}{2}l\{i1+\delta j1\}^j$	$lj\delta^{ij}$
$l^{j+}$	$l^{i1}+\delta^{j1j}$	$l_{j1}^+\delta_{i1}^j$	0	$-l_0\delta^{ji}$	0	$\frac{1}{2}l\{i1\delta j1\}^j$	0	$-lj+\delta^{ij}$
$l^{i2j2}$	$-lj1\{j2\delta^{i2}\}^{i1}$	$-l_{i1}\{i2+\delta_{j1}^{j2}\}$	0	0	$-\frac{1}{2}l\{i2\delta j2\}^i$	0	$L^{i2j2,i1j1}$	$l^i\{i2\delta j2\}^i$
$l^{i2j2+}$	$l^{i1}\{i2+\delta j2\}^{j1}$	$l_{j1}\{j2+\delta_{i1}^{i2}\}$	0	$\frac{1}{2}l\{i2+\delta j2\}$	0	$-L^{i1j1,i2j2}$	0	$-l^i\{i2+\delta j2\}^i$
$g_0^j$	$-F^{i1j1,j}$	$F_{i1j1}^{j+}$	0	$-l^i\delta^{ij}$	$l^{i+}\delta^{ij}$	$-lj\{i1\delta j1\}^j$	$lj\{i1+\delta j1\}^j$	0

Table 1: even-even and odd-even parts of HS symmetry superalgebra  $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ .

We call the superalgebra of the operators  $o_I$  as *half-integer higher-spin symmetry algebra in Minkowski space with a YT having  $k$  rows* and denote it as  $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$ . It appears as the superextension of *integer higher-spin symmetry algebra*  $\mathcal{A}(Y(k), \mathbb{R}^{1,d-1})$  introduced in [8] for bosonic fields.

Hamiltonian analysis of the topological dynamical system of the operators  $\{o_I\}$  permits to classify  $2k(k+1)$  operators  $\{o_a\} = \{t^i, l^{ij}, t^{i1j1}, t_i^+, l_{ij}^+, t_{i1j1}^+\}$  as second-class and  $2(k+1)$  ones  $\{t_0, l_0, l^i, l_j^+\}$  as first-class constraints subsystems whereas  $k$  elements  $g_0^i$  form supermatrix  $\Delta_{ab}(g_0^i)$  in  $\{o_a, o_b\} \sim \Delta_{ab}$ . The subsystem of the second-class constraints  $\{o_a\}$  together with  $\{g_0^i\}$  forms the subalgebra in  $\mathcal{A}(Y(k), \mathbb{R}^{1,d-1})$  to be isomorphic, due to Howe duality, [29], to orthosymplectic  $osp(k|2k)$  algebra (the details, see in [10]).

The HS symmetry superalgebra  $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$  itself can not permit to construct BRST operator with respect to  $o_I$  due to second-class constraints  $\{o_a\}$  presence in it. Therefore we should to convert orthosymplectic algebra  $osp(k|2k)$  of  $\{o_a, g_0^i\}$  into enlarged set of operators  $O_I$  with only first-class constraints.

### 3 Scalar Oscillator realization for $osp(k|2k)$

We consider an additive conversion procedure developed within BRST method, (see e.g. [20]), implying the enlarging of  $o_I$  to  $O_I = o_I + o'_I$ , with additional parts  $o'_I$  supercommuting with all  $o_I$  and determined on a new Fock space  $\mathcal{H}'$ . Now, the elements  $O_I$  are given on  $\mathcal{H} \otimes \mathcal{H}'$  so that a condition for  $O_I$ ,  $[O_I, O_J] \sim O_K$ , leads to the same algebraic relations for  $O_I$  and  $o'_I$  as those for  $o_I$ .

Not going into details of Verma module (special

representation space [30]) construction for the superalgebra  $osp(k|2k)$  of new operators  $o'_I$  considered in [10] and for the case of its  $sp(k|2k)$  subalgebra in [8], we present here their explicit oscillator form in terms of new  $2k(k+1)$  creation and annihilation operators  $(B_d^+, B_d^-) = (f_i^+, b_{ij}^+, d_{rs}^+; f_i, b_{ij}, d_{rs})$ ,  $i, j, r, s = 1, \dots, k; i \leq j; r < s$  as follows (for  $k_0 \equiv l$ )

$$g_0^i = f_i^+ f_i + \sum_{l \leq m} b_{lm}^+ b_{lm} (\delta^{il} + \delta^{im}) + \sum_{r < s} d_{rs}^+ d_{rs} (\delta^{is} - \delta^{ir}) + h^i, \quad (12)$$

$$l_{ij}^{'+} = b_{ij}^+, \quad t_i^{'+} = f_i^+ + 2b_{ii}^+ f_i + 4 \sum_{l=1}^{i-1} b_{li}^+ f_l, \quad (13)$$

$$t_{lm}^{'+} = d_{lm}^+ - \sum_{n=1}^{l-1} d_{nl} d_{nm}^+ - \sum_{n=1}^k (1 + \delta_{nl}) b_{nm}^+ b_{ln}, \quad (14)$$

$$t'_{lm} = - \sum_{n=1}^{l-1} d_{nl}^+ d_{nm} - \sum_{n=1}^k (1 + \delta_{nm}) b_{nl}^+ b_{nm} + \sum_{p=0}^{m-l-1} \sum_{k_1=l+1}^{m-1} \dots \sum_{k_p=l+p}^{m-1} C^{k_p m}(d^+, d) \prod_{j=1}^p d_{k_{j-1} k_j} + [4 \sum_{n=r+1}^{s-1} b_{rn}^+ f_n + (2b_{rr}^+ f_r - f_r^+)] f_s. \quad (15)$$

Note, first, that  $B_c, B_d^+$  satisfy to the standard commutation relations,  $[B_c, B_d^+] = \delta_{cd}$ , for instance,  $\{f_i, f_j^+\} = \delta_{ij}$ , for odd  $f_i, f_j^+$ . Second, the arbitrary parameters  $h^i$  in (12) need to reproduce correct LF for HS field with given spin  $\mathbf{s}$ , whereas the form of the rest elements  $t'_i, l'_{ij}$ , for  $i \leq j$ , to be expressed by means of the operators  $C^{lm}(d^+, d)$ ,  $l < m$ , as well as the prop-

<sup>1</sup>The case of massive HS fields whose system of 2nd-class constraints contains additionally to elements of  $osp(k|2k)$  superalgebra the constraints of isometry subalgebra of Minkowski space  $t_0, l^i, l_i^+, l_0$  may be treated by dimensional reduction of the algebra  $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d})$  for massless HS fields to one  $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$  for massive HS fields, (see [10]). Now, the Dirac equation in (2) is

erty of Hermiticity for them may be found in [8], [10]<sup>1</sup>.

#### 4 BRST-BFV operator and Lagrangian formulations

Due to algebra of  $O_I$  under consideration is a Lie superalgebra  $\mathcal{A}^f(Y(k), \mathbb{R}^{1,d-1})$  the BFV-BRST operator  $Q'$  may be constructed in the standard way<sup>2</sup> as

$$Q' = O_I C^I + \frac{1}{2} C^I C^J f_{JI}^K \mathcal{P}_K (-1)^{\epsilon(O_K) + \epsilon(O_I)} \quad (16)$$

with the constants  $f_{JI}^K$  from the table 1, constraints  $O_I = (T_0, T_i^+, T_i; L_0, L_i^+, L_i, L_{ij}, L_{ij}^+, T_{rs}, T_{rs}^+, G_0^i)$ , fermionic [bosonic] ghost fields and conjugated to them momenta  $(C^I, \mathcal{P}_I) = ((\eta_0, \mathcal{P}_0); (\eta_i^i, \mathcal{P}_i^+); (\eta_i^+, \mathcal{P}_j); (\eta^{ij}, \mathcal{P}_{ij}^+); (\eta_{ij}^+, \mathcal{P}_{ij}); (\vartheta_{rs}, \lambda_{rs}^+); (\vartheta_{rs}^+, \lambda_{rs}); (\eta_G^i, \mathcal{P}_G))$ ,  $[(q_0, p_0), (q_i^+, p_i), (q_i, p_i^+)]$  with the properties

$$\begin{aligned} (\eta^{ij}, \vartheta_{rs}) &= (\eta^{ji}, \vartheta_{rs} \theta^{sr}), \quad \{\vartheta_{rs}, \lambda_{tu}^+\} = \delta_{rt} \delta_{su}, \\ \{\eta_{lm}, \mathcal{P}_{ij}^+\} &= \delta_{li} \delta_{jm}, \quad \{\mathcal{P}_j, \eta_i^+\} = [q_i, p_j^+] = \delta_{ij} \end{aligned} \quad (17)$$

and non-vanishing (anti)commutators  $\{\eta_0, \mathcal{P}_0\} = [q_0, p_0] = \iota$ ,  $\{\eta_G^i, \mathcal{P}_G^j\} = \iota \delta^{ij}$  for zero-mode ghosts<sup>3</sup>.

To construct LF for fermionic HS fields in a  $\mathbb{R}^{1,d-1}$  space-time we partially follow the algorithm of [28], being a particular case of our construction for  $n_3 = 0$ . First, we extract the dependence of  $Q'$  (16) on the ghosts  $\eta_G^i, \mathcal{P}_G^i$ , to obtain generalized spin operator  $\sigma^i$  and the BRST operator  $Q$  only for the system of converted first-class constraints  $\{O_I\} \setminus \{G_0^i\}$  on appropriate Hilbert subspaces:

$$Q' = Q + \eta_G^i (\sigma^i + h^i) + \mathcal{B}^i \mathcal{P}_G^i, \quad \text{with some } \mathcal{B}^i, \quad (18)$$

$$\begin{aligned} Q &= \left( \frac{1}{2} q_0 T_0 + \frac{1}{2} \eta_0 L_0 + q_i^+ T^i \eta_i^+ L^i + \sum_{l \leq m} \eta_{lm}^+ L^{lm} \right. \\ &\quad \left. + \sum_{l < m} \vartheta_{lm}^+ T^{lm} + h.c. \right) + \frac{1}{2} \widehat{C}^I \widehat{C}^J f_{JI}^K \widehat{\mathcal{P}}_K, \end{aligned} \quad (19)$$

$$\begin{aligned} \sigma^i &= G_0^i - h^i - \eta_i \mathcal{P}_i^+ + \eta_i^+ \mathcal{P}_i + q_i p_i^+ + q_i^+ p_i \\ &\quad + \sum_m (1 + \delta_{im}) (\eta_{im}^+ \mathcal{P}^{im} - \eta_{im} \mathcal{P}_{im}^+) \\ &\quad + \sum_{l < i} [\vartheta_{li}^+ \lambda^{li} - \vartheta^{li} \lambda_{li}^+] - \sum_{i < l} [\vartheta_{il}^+ \lambda^{il} - \vartheta^{il} \lambda_{il}^+], \end{aligned} \quad (20)$$

where  $\{\widehat{C}^I, \widehat{\mathcal{P}}^I\} \equiv \{C^I, \mathcal{P}^I\} \setminus \{\eta_G^i, \mathcal{P}_G^i\}$ . Next, we choose a representation of  $\mathcal{H}_{tot}$ :  $(q_i, p_i, \eta_i, \eta_{ij}, \vartheta_{rs}, \mathcal{P}_0, p_0, \mathcal{P}_i, \mathcal{P}_{ij}, \lambda_{rs}, \mathcal{P}_G^i) |0\rangle = 0$ , and suppose that the field vectors  $|\chi\rangle$  as well as the gauge parameters  $|\Lambda\rangle$  do not

changed on massive equation corresponding to the constraint  $t_0$  ( $t_0 = -i\tilde{\gamma}^\mu \partial_\mu + \tilde{\gamma} m$ ) acting on the same *basic* vector  $|\Psi\rangle$  (5).

<sup>2</sup>for construction of BRST operator for quadratic HS symmetry algebra for mixed-symmetric tensors on AdS subject to YT with 2 rows, see [23].

<sup>3</sup>The ghosts possess the standard ghost number distribution,  $gh(C^I) = -gh(\mathcal{P}_I) = 1 \implies gh(Q') = 1$ .

<sup>4</sup>The brackets  $(n)_{fi}, (n)_{pj}, (n)_{ij}$  in (21) means, e.g., for  $(n)_{ij}$  the set of indices  $(n_{11}, \dots, n_{1k}, \dots, n_{k1}, \dots, n_{kk})$ . The sum above is taken over  $n_{b0}, n_{ae}, n_{bg}, n_{ij}, p_{rs}$  and running from 0 to infinity, and over the rest  $n$ 's from 0 to 1.

depend on ghosts  $\eta_G^i$ :

$$\begin{aligned} |\chi\rangle &= \sum_n \prod_{l, i \leq j, r < s}^k (f_l^+)^{n_l^0} (b_{ij}^+)^{n_{ij}} (d_{rs}^+)^{p_{rs}} q_0^{n_{b0}} \eta_0^{n_{f0}} \\ &\quad \prod_{e, g, i, j, l \leq m, n \leq o} (q_e^+)^{n_{ae}} (p_g^+)^{n_{bg}} (\eta_i^+)^{n_{fi}} (\mathcal{P}_j^+)^{n_{pj}} (\eta_{lm}^+)^{n_{flm}} \\ &\quad (\mathcal{P}_{no}^+)^{n_{pno}} \prod_{r < s, t < u} (\vartheta_{rs}^+)^{n_{frs}} (\lambda_{tu}^+)^{n_{\lambda tu}} \times \quad (21) \\ |\Psi(a_i^+)&= \Psi_{(n)_{fi}^0(n)_{ij}(p)_{rs}}^{n_{b0} n_{f0} (n)_{ae}(n)_{bg}(n)_{fi}(n)_{pj}(n)_{flm}(n)_{pnd}(n)_{frs}(n)_{\lambda tu}} \rangle^4. \end{aligned}$$

We denote by  $|\chi^k\rangle$  the state (21) satisfying to  $gh(|\chi^k\rangle) = -k$ . Thus, the physical state having the ghost number zero is  $|\chi^0\rangle$ , the gauge parameters  $|\Lambda\rangle$  having the ghost number  $-1$  is  $|\Lambda^0\rangle$  and so on. The vector  $|\chi^0\rangle$  must contain physical string-like vector  $|\Psi\rangle = |\Psi(a_i^+)^{(0)_{fo} \dots (0)_{rs}}\rangle$ :

$$|\chi^0\rangle = |\Psi\rangle + |\Psi_A\rangle : |\Psi_A\rangle|_{[B_a^+ = C^I = \mathcal{P}_I = 0]} = 0. \quad (22)$$

Independence of the vectors (21) on  $\eta_G^i$  transforms the equation for the physical state  $Q'|\chi^0\rangle = 0$  and the BRST complex of the reducible gauge transformations,  $\delta|\chi\rangle = Q'|\Lambda^0\rangle$ ,  $\delta|\Lambda^0\rangle = Q'|\Lambda^1\rangle$ ,  $\dots$ ,  $\delta|\Lambda^{(r-1)}\rangle = Q'|\Lambda^{(r)}\rangle$ , to the relations:

$$\begin{aligned} (Q|\chi^0\rangle, \delta|\chi^0\rangle, \dots, \delta|\Lambda^{(r-1)}\rangle) &= (0, Q|\Lambda^0\rangle, \dots, Q|\Lambda^{(r)}\rangle), \\ [\sigma^i + h^i] (|\chi^0\rangle, |\Lambda^0\rangle, \dots, |\Lambda^{(r)}\rangle) &= 0, \end{aligned} \quad (23)$$

with  $r = \sum_{l=1}^k n_l + k(k-1)/2 - 1$  being the stage of reducibility both for massless and for the massive fermionic HS field. Resolution the spectral problem from the Eqs.(23) yields the eigenvectors of the operators  $\sigma^i$ :  $|\chi^0\rangle_{(n)_k}, |\Lambda^0\rangle_{(n)_k}, \dots, |\Lambda^r\rangle_{(n)_k}$ ,  $n_1 \geq n_2 \geq \dots n_k \geq 0$  and corresponding eigenvalues of the parameters  $h^i$  (for massless HS fields and  $i = 1, \dots, k$ ),

$$-h^i = m_i + \frac{d-4i}{2}, \quad m_1, \dots, m_{k-1} \in \mathbb{Z}, m_k \in \mathbb{N}_0. \quad (24)$$

One can show, first, the operator  $Q$  is nilpotent on the subspaces determined by the solution for the Eq. (23), second, to construct Lagrangian for the field corresponding to a definite YT (1) we must put  $m_i = n_i$ , and, third, one should substitute  $h^i$  corresponding to the chosen  $n_i$  (24) into  $Q$  (18) and relations (23).

To get the Lagrangian formulation with only first order derivatives, we, because of the functional dependence of the operator  $L_0$  on fermionic one  $T_0$ ,  $L_0 = -T_0^2$ , may to gauge away a dependence on  $L_0, \eta_0$  from the BRST operator  $Q$  (19) and from the whole set

of the vectors  $|\chi^0\rangle_{(n)_k}$ ,  $|\Lambda^{(s)}\rangle_{(n)_k}$ . To do so, we extract the zero-mode ghosts from the operator  $Q$  as:

$$Q = q_0 \tilde{T}_0 + \eta_0 L_0 + \imath(\eta_i^+ q_i - \eta_i q_i^+) p_0 - \imath(q_0^2 - \eta_i^+ \eta_i) \mathcal{P}_0 + \Delta Q, \quad (25)$$

where the explicit form of  $\Delta Q$  is easily restored from Eqs. (18), (25) and

$$\tilde{T}_0 = T_0 - 2q_i^+ \mathcal{P}_i - 2q_i \mathcal{P}_i^+ : \tilde{T}_0^2 = -L_0. \quad (26)$$

We also expand the state vector and gauge parameters in powers of the zero-mode ghosts, for  $s = 0, \dots, \sum_{o=1}^k n_o + k(k-1)/2 - 1$ ,  $m = 0, 1$ :

$$|\chi\rangle = \sum_{l \geq 0} q_0^l (|\chi_0^l\rangle + \eta_0 |\chi_1^l\rangle), gh(|\chi_m^l\rangle) = -(m+l) \quad (27)$$

$$|\Lambda^{(s)}\rangle = \sum_{l=0}^k q_0^l (|\Lambda^{(s)l}_0\rangle + \eta_0 |\Lambda^{(s)l}_1\rangle). \quad (28)$$

Now, we may gauge away of all the fields and gauge parameters by means of the equations of motion and set of the gauge transformations (23) except two,  $|\chi_0^0\rangle$ ,  $|\chi_0^1\rangle$  for the fields and  $|\Lambda^{(s)l}_0\rangle$ , for  $l = 0, 1$  and  $s = 0, \dots, r$ , for the gauge parameters. To do so, we use in part the procedure described in [21], [28].

As the result, the first-order equations of motion corresponding to the field with given spin  $(n_1 + \frac{1}{2}, \dots, n_k + \frac{1}{2})$  have the form in terms of the matrix notations,

$$\begin{pmatrix} \tilde{T}_0 & \Delta Q \\ \Delta Q & \frac{1}{2} \{ \tilde{T}_0, \eta_i^+ \eta_i \} \end{pmatrix} \begin{pmatrix} |\chi_0^0\rangle_{(n)_k} \\ |\chi_0^1\rangle_{(n)_k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (29)$$

They are Lagrangian ones and can be deduced from the Lagrangian action for fixed spin  $(m)_k = (n)_k$ , (being standardly defined up to an overall factor and with omitting the subscript  $(n)_k$ )

$$\mathcal{S}_{(n)_k} = \langle \tilde{\chi}_0^0 |, \langle \tilde{\chi}_0^1 | K \begin{pmatrix} \tilde{T}_0 & \Delta Q \\ \Delta Q & \frac{1}{2} \{ \tilde{T}_0, \eta_i^+ \eta_i \} \end{pmatrix} \begin{pmatrix} |\chi_0^0\rangle \\ |\chi_0^1\rangle \end{pmatrix}, \quad (30)$$

where the standard odd scalar product for the creation and annihilation operators in  $\mathcal{H}_{tot} = \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H}_{gh}$  is assumed and non-degenerate operator  $K = K_{(n)_k}$  provides reality of the action following from modifying Hermiticity for  $\sigma'_I$  in Section 3. The action (30) is invariant with respect to the gauge transformations, following from the tower of the Eqs. (23) with omitting  $(n)_k$ ,

$$\delta \begin{pmatrix} |\Lambda_0^{(s)0}\rangle \\ |\Lambda_0^{(s)1}\rangle \end{pmatrix} = \begin{pmatrix} \tilde{T}_0 & \Delta Q \\ \Delta Q & \frac{1}{2} \{ \tilde{T}_0, \eta_i^+ \eta_i \} \end{pmatrix} \begin{pmatrix} |\Lambda_0^{(s+1)0}\rangle \\ |\Lambda_0^{(s+1)1}\rangle \end{pmatrix}, \quad (31)$$

for  $s = -1, 0, \dots, \sum_{o=1}^k n_o + k(k-1)/2 - 1$ , and  $|\Lambda_0^{(-1)l}\rangle \equiv |\chi_0^l\rangle$ .

Concluding, one can prove the action (30) indeed reproduces the basic conditions (2)–(4) for massless (massive) HS fields. General action (30) gives, in principle, a straight receipt to obtain the Lagrangian for any component field from general vectors  $|\chi_0^l\rangle_{(n)_k}$ .

## 5 Conclusion

Thus, we have constructed a gauge-invariant unconstrained Lagrangian description of free half-integer HS fields belonging to an irreducible representation of the Poincare group  $ISO(1, d-1)$  with the arbitrary YT having  $k$  rows in the “metric-like” formulation. The results of this study are the general and obtained on the base of universal method which is applied by the unique way to both massive and massless bosonic HS fields with a mixed symmetry in a Minkowski space of any dimension.

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